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# FEATURES OF THE PRESSURE–ATTENUATION CURVE IN RELAXATION FILTRATION OF A FLUID

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Laboratory experiments have shown that, for fluid filtration processes with a characteristic fluctuation time of ~$10^3$ sec, theoretical predictions based on a model of the elastic regime can differ from observed quantities by an order of magnitude [1-3]. Therefore, in describing rapidly varying fluid filtration phenomena, the classic elastic equations [4, 5] must be avoided, and equations from the relaxation theory of filtration [6, 7] must be used instead, in particular, for the initial section of the pressure–attenuation curve. In earlier approximate formulas for the pressure–attenuation curve, the relaxation kernel had a somewhat special form [6]. The most general case [6] corresponds to a vibrating Fourier-type relaxation kernel in the form of a ratio of two second-order polynomials. In this work exact results are found for the initial section of the pressure–attenuation curve for an arbitrary kernel which is consistent with physical and thermodynamic requirements.

1. We examine a homogeneous porous medium which is saturated with fluid. Isothermal processes are studied in which the fluid density $\rho$ differs only slightly from some fixed value $\rho_0$; therefore a linear expression can be used for the pressure

$$p = p_0 + E(\rho - \rho_0)/\rho_0. \tag{1.1}$$

In the relaxation theory of filtration [6, 7], Darcy's law is generalized as follows:

$$\mathbf{u}(t_0, r) = -k\mu^{-1} \int_{-\infty}^{+\infty} K(t_0 - t)\nabla G(t, \mathbf{r})\, dt, \quad G = p + \rho\varphi. \tag{1.2}$$

Here $\mathbf{u}$ is the filtration velocity; k is the permeability; $\varphi$ is the gravitational potential; and $\mu$ is the viscosity, which will be considered constant. The kernel K = K(t), which does not depend on the spatial coordinates, characterizes the internal relaxation processes in the system of the porous medium and the fluid. The function K = K(t) satisfies a series of conditions which follow from physical and thermodynamic considerations [2]:

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1. $K = K(t)$ is a nonnegative function with dimensions of inverse time.

2. $\int_{-\infty}^{+\infty} K(t)\,dt = 1.$

3. The carrier of the function $K = K(t)$ lies on the axis $[0, +\infty)$. On this axis $K = K(t)$ is a smooth, monotonic, rapidly decaying function. The condition $K(0) < +\infty$ guarantees a finite velocity of signal propagation during filtration [8].

Hereafter, the symbol $f_F$ denotes the Fourier transform of any function of time $f = f(t)$:

$$f_F(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega t} f(t)\,dt, \quad \omega \in R.$$

According to the Paley–Wiener theorem, it follows from condition 3 that $K_F = K_F(\omega)$ can be continued analytically into the lower half of the complex plane [9, 10]. According to condition 2, $K_F(0) = 1$. There is also the thermodynamic condition:

4. $\mathrm{Re} K_F(\omega) > 0$, $\omega \in R$.

For large $|\omega|$, the following expansion is valid

$$K_F(\omega) = K(0)(i\omega)^{-1} + K'(0)(i\omega)^{-2} + O(\omega^{-3}) \, . \tag{1.3}$$

From (1.3) and condition 4 we require that $K'(0) < 0$. Furthermore, from condition 4, Eq. (1.3), and the general theory [11], it follows that the holomorphic function $K_F = K_F(\omega)$ has no zeros for $\mathrm{Im}\,\omega < 0$, therefore it reflects the half-plane $\mathrm{Im}\,\omega < 0$ into itself. In particular, the strict inequality 4 is observed over the whole lower complex half-plane.

During filtration of a fluid in a porous medium, the continuity equation $\partial(m\rho)/\partial t + \mathrm{div}\,(\rho \mathbf{u}) = 0$ is obeyed ($m$ is the porosity). This question, plus (1.1) and (1.2), gives an equation for determining the dynamic pressure:

$$\frac{\partial p}{\partial t}(t_0, \mathbf{r}) = \varkappa \int_{-\infty}^{+\infty} K(t_0 - t)\,\Delta p(t, \mathbf{r})\,dt, \; \varkappa = \frac{kE}{m\mu}, \tag{1.4}$$

where $\Delta$ is the Laplacian operator.

We will examine the two-dimensional problem of operating a well with a variable output. In this case $p = p(t, r)$, $0 < r_1 \le r \le r_2$, where $r_1$ is the radius of the well and $r_2$ is the radius of the recharge contour. Equation (1.4) takes the form

$$\frac{\partial p}{\partial t}(t_0, r) = \varkappa \int_{-\infty}^{+\infty} K(t_0 - t)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) p(t, r)\,dt. \tag{1.5}$$

The boundary conditions are constant pressure at the recharge contour:

$$p(t, r_2) = p_0 = \mathrm{const} \tag{1.6}$$

and a given output $q = q(t)$ per unit productive thickness of the bed:

$$q(t_0) = \lambda \int_{-\infty}^{+\infty} K(t_0 - t)\frac{\partial}{\partial r} p(t, r_1)\,dt, \quad \lambda = 2\pi r_1 k \mu^{-1}\rho_0. \tag{1.7}$$

The condition (1.7) is obtained from the relaxation law of filtration (1.2).

In order to simplify future formulas, we choose a system of units for the time and length variables in which $\kappa = r_1 = 1$. We set $P = p - p_0$. Then the equation for $P_F = P_F(\omega, r)$ follows from (1.5)-(1.7):

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{i\omega}{K_F(\omega)}\right) P_F = 0 \tag{1.8}$$

and the boundary conditions are

$$P_F|_{r=r_2}=0 \text{ and } \left.\frac{\partial P_F}{\partial r}\right|_{r=1}=\lambda^{-1}q_F/K_F. \tag{1.9}$$

We determine the function $\alpha = \alpha(\omega)$ from the relationships

$$\alpha^2 = i\omega/K_F(\omega), \ \mathrm{Re}\alpha \geqslant 0. \tag{1.10}$$

It turns out that $\alpha = \alpha(\omega)$ is homomorphic in the lower complex half-plane of the function and is continuous all the way to the real axis. Actually,

$$\mathrm{Im}(i\omega/K_F) = (\mathrm{Re}\omega\mathrm{Re}K_F + \mathrm{Im}\omega\mathrm{Im}K_F)/|K_F|^2 \tag{1.11}$$

and

$$\mathrm{Im}\,K_F = -\int_0^{+\infty} e^{t\,\mathrm{Im}\,\omega}\sin(t\,\mathrm{Re}\,\omega)\,K(t)\,dt. \tag{1.12}$$

As a result of condition 3, Eq. (1.12) yields the inequality $\mathrm{Re}\omega\mathrm{Im}K_F \leq 0$. Therefore, from (1.11) and the assumptions, it follows that $\mathrm{Im}(i\omega/K_F) = 0$, only if $\mathrm{Re}\,\omega = 0$. In the last case, however, if $\omega \neq 0$, then $i\omega/K_F > 0$.

Therefore, for $\mathrm{Im}\,\omega \leq 0$ and $\omega \neq 0$, we have $\mathrm{Re}\alpha(\omega) > 0$. Thus, Eq. (1.10) gives $\alpha = \alpha(\omega)$ as a smooth single-valued function. Generally speaking, the function $\alpha = \alpha(\omega)$ can be continued analytically into the upper half of the complex plane, but then it will have a cut along the imaginary axis because of the cut related to the branching of the square root and the singularities in $K_F$.

There is a simple solution to the problem (1.8) and (1.9)

$$P_F = \frac{q_F(-I_0(\alpha r_2)K_0(\alpha r)+K_0(\alpha r_2)I_0(\alpha r))}{\lambda K_F\alpha(K_0(\alpha r_2)I_1(\alpha)+K_1(\alpha)I_0(\alpha r_2))}, \tag{1.13}$$

where $I_\nu(z)$, $K_\nu(z)$ are the MacDonald functions [12]. Here $I_\nu(z)$ is a complete function, but $K_\nu(z)$ has a cut along the negative real axis.

We now examine the asymptotic expansions of $\alpha$ and $P_F$ for $|\omega| \to +\infty$. From (1.3) and (1.10) we find

$$\alpha = i\omega a_1 + a_0 + O(\omega^{-1}), \quad a_1 = (K(O))^{-1/2}, \quad a_0 = -\frac{1}{2}K'(O)a_1^3. \tag{1.14}$$

From (1.13), (1.14), and the asymptotic expansions of the MacDonald functions [12, 13], we obtain

$$P_F/q_F = -a_1\lambda^{-1}(c(\omega,\ r) - c^{-1}(\omega,\ r))/(c(\omega,\ 1) + c^{-1}(\omega,\ 1)) + o(1),$$
$$c(\omega,\ r) = \exp\left[(i\omega a_1 + a_0)(r_2 - r)\right].$$

Thus, the convergence of the integral

$$P(t,r) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\omega t}q_F(\omega)\,P_F(\omega,r)\,d\omega \tag{1.15}$$

depends essentially on the properties of $q_F$. For $\omega \to 0$, we use (1.13) and the expressions for the MacDonald functions [12, 13] to find

$$P_F/q_F = \lambda^{-1}\ln(r/r_2) + o(1). \tag{1.16}$$

If $q(t) = Q = \mathrm{const}$, then from (1.15), (1.16), and the formula $q_F(\omega) = 2\pi Q\delta(\omega)$ we find the exact solution to be the same as for elastic theory [4, 5]:

$$P = \lambda^{-1}Q\ln(r/r_2). \tag{1.17}$$

2. As in the classic formulation of the pressure–attenuation curve problem, we let $q = Q\cdot\theta(-t)$, where $Q$ is a constant and $\theta(t)$ is the Heaviside function. Then for $t < 0$,

P is given by (1.17). We now change the notation. We set $P(t, r) = p(t, r) - \lambda^{-1}\ln(r/r_2) - p_0$. Then $P = 0$ for $t < 0$. From the linearity of the problem (1.5)-(1.7), P can be computed from Eqs. (1.13) and (1.15), where $q_F = Qi/(\omega - i\varepsilon)$, which corresponds to $q(t) = -Q\cdot\theta(t)$. Here $\varepsilon$ is a small positive quantity, which must be set to zero after the calculations are complete.

Because in this model the velocity of signal propagation is finite [8] and we are interested in P(t, r) for small values of the arguments, the dependence on $r_2$ should be insignificant and we can extend $r_2$ to infinity in (1.13). Then by using the asymptotic MacDonald functions [12, 13], we obtain

$$P_F = -QiK_0(\alpha r)/[\lambda K_F \alpha K_1(\alpha)(\omega - i\varepsilon)]. \tag{2.1}$$

We will investigate the pressure change in the well $F(t) = P|_{r=1}$. From (1.15) and (2.1) we have

$$F(t) = -\frac{Qi}{2\pi\lambda}\int_{-\infty}^{+\infty}\frac{e^{i\omega t}f(\omega)\,d\omega}{\omega - i\varepsilon}, \quad f(\omega) = \frac{K_0(\alpha)}{K_F\alpha K_1(\alpha)}.$$

From the preceding it follows that the function $f = f(\omega)$ is holomorphic in the half-plane $\mathrm{Im}\,\omega < 0$. We apply the asymptotic MacDonald functions [12, 13] and the expansions (1.3) and (1.14), and compute the asymptotic expansions of $f(\omega)$ for small and large $\omega$:

$$\omega \to 0, \quad f(\omega) = \frac{1}{2}\ln(i\omega) + \ln(\gamma/2) + o(1), \quad \gamma = e^C \tag{2.2}$$

and

$$|\omega| \to +\infty, \quad f(\omega) = a_1 + i\nu\omega^{-1} + O(\omega^{-2}), \quad \nu = a_1^{-1}(2 - a_0)/i; \tag{2.3}$$

where C is Euler's constant. We define the function $h_1 = h_1(\omega)$ from the formula

$$h_1(\omega) = \frac{\ln(i\omega)}{2(\omega^2+1)} + \frac{ix_1}{\omega - iy_1} + \frac{ix_2}{\omega - iy_2} + a_1,$$

where the real numbers $x_1$, $x_2$, $y_1$, and $y_2$ are the solutions of the (complex) equations and the inequalities

$$x_1 + x_2 = \nu, \quad -x_1/y_1 - x_2/y_2 + a_1 = \ln(\gamma/2), \quad y_1,\ y_2 > 0. \tag{2.4}$$

We set $h_2 = f - h_1$. Then $F(t) = H_1(t) + H_2(t)$, where

$$H_1(t) = -\frac{Qi}{2\pi\lambda}\int_{-\infty}^{+\infty}\frac{e^{i\omega t}h_1(\omega)\,d\omega}{\omega - i\varepsilon}; \quad H_2(t) = -\frac{Qi}{2\pi\lambda}\int_{-\infty}^{+\infty}e^{i\omega t}\omega^{-1}h_2(\omega)\,d\omega.$$

According to (2.2)-(2.4), the function $h_2(\omega)/\omega$ has the following properties: it is holomorphic for $\mathrm{Im}\,\omega < 0$; it is smooth outside the point $\omega = 0$, where it has an integrable (logarithmic) singularity; for large $|\omega|$ it behaves asymptotically as $h_2(\omega)/\omega = O(1/\omega^3)$. Using the Lebesgue theorem on the transition to the limit under the integral, it is easy to be convinced that the function $H_2 = H_2(t)$ is continuous and differentiable for all t. Because it is obvious that $H_2(t) = 0$ for $t < 0$ (the Paley–Wiener theorem [9, 10]), $H_2(0) = H'_2(0) - 0$. From this one easily can derive that $F(t) = H_1(t) + o(t)$. In particular

$$F|_{t=+0} = H_1|_{t=+0}, \quad \left.\frac{dF}{dt}\right|_{t=+0} = \left.\frac{dH_1}{dt}\right|_{t=+0}. \tag{2.5}$$

For computing the function $H_1 = H_1(t)$, we use formulas 3.352.6, 3.352.4, 8.214.1, and 8.214.2, respectively, from [13]:

$$\mathrm{V.p.}\int_0^{+\infty}\frac{e^{-bz}\,dz}{a - z} = e^{-ab}\mathrm{Ei}(ab) \qquad (a > 0,\ \mathrm{Re}\,b > 0); \tag{2.6}$$

$$\int_0^{+\infty} \frac{e^{-bz}\,dz}{z+a} = -e^{ab}\,\mathrm{Ei}(-ab) \qquad (|\arg a| < \pi,\ \mathrm{Re}\, b > 0); \tag{2.7}$$

$$\mathrm{Ei}(z) = C + \ln(-z) + \sum_{k=1}^{+\infty} \frac{z^k}{kk!} \qquad (z < 0); \tag{2.8}$$

$$\mathrm{Ei}(z) = C + \ln z + \sum_{k=1}^{+\infty} \frac{z^k}{kk!} \qquad (z > 0) \tag{2.9}$$

where Ei(z) is the exponential integral function [13]. Now we compute the auxiliary integrals

$$J_1 = \int_{-\infty}^{+\infty} \frac{e^{i\omega t}\ln(i\omega)\,d\omega}{\omega - ai}, \quad J_2 = \int_{-\infty}^{+\infty} \frac{e^{i\omega t}\ln(i\omega)\,d\omega}{\omega + ai}, \quad a > 0, \quad t > 0, \tag{2.10}$$

keeping in mind that $\ln z$ is an analytic function with a cut along the negative real axis, which in the ω-plane corresponds to the positive imaginary axis. Shifting the integration contour in (2.10) to avoid the cut, we use (2.6) and (2.7) and obtain

$$J_1 = 2\pi i e^{-at}(\ln a - \mathrm{Ei}(at)),\ J_2 = -2\pi i e^{at}\mathrm{Ei}(-at). \tag{2.11}$$

Now, we use the expansion

$$(\omega - i\varepsilon)^{-1}(\omega^2 + 1)^{-1} = A(\omega - i\varepsilon)^{-1} + B(\omega - i)^{-1} + C(\omega + i)^{-1},$$
$$A = (1 - \varepsilon^2)^{-1},\ B = -2^{-1}(1 - \varepsilon)^{-1},\ C = -2^{-1}(1 + \varepsilon)^{-1},$$

and also Eq. (2.11) and the theorem of residues, and easily calculate $H_1(t)$. Here it is convenient to go to the limit $\varepsilon \to +0$ using (2.9). Then

$$H_1(t) = Q\lambda^{-1}\Big\{\frac{1}{4}\left(e^t\,\mathrm{Ei}(-t) + e^{-t}\,\mathrm{Ei}(t) - 2C - 2\ln t\right) +$$
$$+ \frac{x_1}{y_1}\left(e^{-y_1 t} - 1\right) + \frac{x_2}{y_2}\left(e^{-y_2 t} - 1\right) + a_1\Big\}.$$

From this equation, (2.5), (2.8), and (2.9) we find

$$F|_{t=+0} = Q\lambda^{-1}a_1,\ dF/dt|_{t=+0} = -Q\lambda^{-1}(x_1 + x_2) = -Q\lambda^{-1}\nu. \tag{2.12}$$

3. Equations (2.12), which are the basic result of this analysis, physically mean that, after the well is stopped, the pressure undergoes a drop, and then begins to grow with a finite slope. In dimensional variables, Eq. (2.12) takes the form

$$F|_{t=+0} = Q\lambda^{-1}\varkappa^{1/2}a_1,\quad \left.\frac{dF}{dt}\right|_{t=+0} = -Q\lambda^{-1}\varkappa^{1/2}a_1^{-1}\left(2 - a_0\varkappa^{3/2}r_1^{-3}\right). \tag{3.1}$$

We will investigate how the observed effect behaves in the transition to the elastic model. We set $K(t) = f(t/\varepsilon)/\varepsilon$, where $f(t)$ is a smooth positive function for $t > 0$, which is normalized to $\int_0^{+\infty} f(t)\,dt = 1$, where ε is a small positive parameter. As $\varepsilon \to 0$, $K(t)$ tends to the Dirac δ function, and the model (1.2) transforms to the elastic model. Then, it is easy to see that

$$a_1 = \varepsilon^{1/2}(f(0))^{-1/2},\ a_0 = \varepsilon^{-1/2}f'(0)(f(0))^{-3/2},$$
$$F|_{t=+0} \to 0,\ dF/dt|_{t=+0} \to +\infty.$$

Thus, the finiteness of quantities in (3.1) is a specific property of the relaxation model.

We note in conclusion that a pressure jump can be explained as follows. In the relaxation model, a perturbation in the fluid density propagates with a velocity $v = [\kappa K(0)]^{1/2}$ [8]. In the problem, which actually was studied in Sec. 2, there was pumping into the bed, with an outflow Q per unit bed thickness. During a time $\Delta t$ a mass $Q\Delta t$ was pumped into the bed, which corresponds to a density increase of $\Delta\rho$ in a volume $(2\pi r_1 v\Delta t)$ of the porous medium. It is easy to see that the relationship $Q\Delta t = 2\pi r_1 vm\Delta t\Delta\rho$ is equivalent to the first of Eqs. (3.1).

This method can be used to compute the initial section of the pressure—attenuation curve to any accuracy.

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